

ON CALABI'S DIASTASIS FUNCTION OF THE CIGAR METRIC

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ABSTRACT. We show that the Cigar metric on \mathbb{C} is an example of real analytic Kähler manifold with globally defined and positive Calabi's diastasis function which cannot be Kähler immersed into any (finite or infinite dimensional) complex space form.

1. INTRODUCTION

In his seminal paper [3], E. Calabi provides a criterion for a Kähler manifold (M, g) to admit a holomorphic and isometric (from now on Kähler) immersion into a complex space form, finite or infinite dimensional. Recall that a complex space form, that we assume to be complete and simply connected, up to homotheties can be of three types, according to the sign of the constant holomorphic sectional curvature:

- (i) the complex Euclidean space \mathbb{C}^N of complex dimension $N \leq \infty$, endowed with the flat metric. Here \mathbb{C}^∞ denotes the Hilbert space $\ell^2(\mathbb{C})$ consisting of sequences w_j , $j = 1, 2, \dots$, $w_j \in \mathbb{C}$ such that $\sum_{j=1}^{+\infty} |w_j|^2 < +\infty$.
- (ii) the complex projective space \mathbb{CP}^N of complex dimension $N \leq \infty$, with the Fubini-Study metric g_{FS} .
- (iii) the complex hyperbolic space \mathbb{CH}^N of complex dimension $N \leq \infty$, that is the unit ball $B \subset \mathbb{C}^N$ given by $B = \left\{ (z_1, \dots, z_N) \in \mathbb{C}^N, \sum_{j=1}^N |z_j|^2 < 1 \right\}$ endowed with the hyperbolic metric.

The criterion Calabi states is given in terms of the *diastasis function* associated to the metric g , defined as follows. By duplicating the variables z and \bar{z} , a Kähler potential Φ around $p \in M$ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in M \times M$. The diastasis function is defined by the formula:

$$(1) \quad D^g(p, q) = \tilde{\Phi}(p, \bar{p}) + \tilde{\Phi}(q, \bar{q}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}), \quad (p, q) \in U.$$

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Once p is fixed and z are coordinates around it, $D^g(p, q) = D_p^g(z)$ is a Kähler potential in a neighborhood of p . The existence of a local Kähler immersion into a complex space form depends only on the derivatives with respect to z and \bar{z} of $D_p^g(z)$, evaluated at p . In particular, when (M, g) is a complex curve endowed with a radial metric g and the ambient space is the complex projective space, the criterion reads as follows:

Calabi's criterion. *Let (M, g) be a real analytic Kähler curve such that in a coordinate system $\{z\}$ centered at $p \in M$, $D_p^g(z)$ depends only on $|z|^2$. Then (M, g) admits a local Kähler immersion into $\mathbb{CP}^{N \leq \infty}$ iff for any $n > 0$:*

$$\frac{\partial^{2n} \exp(D_0(|z|^2))}{\partial z^n \partial \bar{z}^n} \Big|_0 \geq 0.$$

Further, the image of M is not contained in any totally geodesic submanifold of the N -dimensional ambient space iff the number of derivatives in the above inequality different from zero are exactly N .

A local Kähler immersion can be extended to a global one iff for each point $p \in M$, the maximal analytical extension of $D_p^g(z)$ is single valued, condition which is fulfilled when M is simply connected.

In this paper, we are interested in studying how the multiplication of the Kähler metric g by a positive constant c affects the existence of a (local) Kähler immersion of (M, cg) into complex space forms.

Observe that when one studies Kähler immersions into the complex Euclidean space, the multiplication of the metric g by c is harmless. In fact, if $f : M \rightarrow \mathbb{C}^N$, $N \leq \infty$, satisfies $f^*(g_0) = g$ then $(\sqrt{c}f)^*(g_0) = cg$.

A very interesting example in this sense is given by Cartan domains Ω , i.e. irreducible bounded homogeneous domains of \mathbb{C}^d endowed with their Bergman metric g_B . The existence of a Kähler immersion of (Ω, cg_B) into \mathbb{CP}^∞ depends firmly on c and it is strictly related to the Wallach set $W(\Omega)$ of Ω (see [12]). More precisely, if γ is the genus of Ω , then (Ω, cg_B) , $c > 0$ admits a Kähler immersion into \mathbb{CP}^∞ if and only if $c\gamma$ belongs to $W(\Omega) \setminus \{0\}$. Recall that $W(\Omega)$ is a subset of the real line depending on two of the domain's invariants, denoted by a (strictly positive real number) and r (the rank of Ω), and it is given by:

$$W(\Omega) = \left\{ 0, \frac{a}{2}, 2\frac{a}{2}, \dots, (r-1)\frac{a}{2} \right\} \cup \left((r-1)\frac{a}{2}, \infty \right).$$

(the author is referred to [1], [7] and [15] for more details and results about the Wallach set). This result has been generalized by the first author and R. Mossa [11] for bounded homogeneous domains.

In view of the case of Hermitian symmetric spaces it is natural and interesting to exhibit examples of manifolds (M, cg) that do not admit a Kähler immersion into any complex space form for any value of $c > 0$. At the end of his paper Calabi himself provides the following two examples (in the first one M is compact and in the second one M is noncompact and complete).

Example 1. Consider the product $\mathbb{CP}^1 \times \mathbb{CP}^1$ endowed with the metric $g = b_1 g_{FS} \oplus b_2 g_{FS}$, with b_1, b_2 positive real numbers such that b_2/b_1 is irrational. Then $(\mathbb{CP}^1 \times \mathbb{CP}^1, cg)$ does not admit a Kähler immersion into \mathbb{CP}^∞ for any value of c . In fact, in [3, Th.13], Calabi proves that (\mathbb{CP}^n, cg_{FS}) admits a Kähler immersion into \mathbb{CP}^∞ iff $1/c$ is a positive integer, and this property cannot be fulfilled by both $1/cb_1$ and $1/cb_2$.

Example 2. Consider on \mathbb{C} the metric g whose associate Kähler form ω is given by: $\omega = (4 \cos(z - \bar{z}) - 1) dz \wedge d\bar{z}$. A Kähler immersion of (\mathbb{C}, cg) into \mathbb{CP}^∞ is not possible since the diastasis:

$$D(p, q) = 4 [\cos(p - \bar{p}) + \cos(q - \bar{q}) - \cos(p - \bar{q}) - \cos(q - \bar{p})] - |p - q|^2,$$

takes negative values, e.g. for $q = p + 2\pi$.

These two examples suggest a refinement of the previous problem. In fact, both the metrics described present geometrical obstructions to the existence of a Kähler immersion into \mathbb{CP}^∞ that put aside the role of c . More precisely, in the first example, the Kähler form ω associated to g is not integral, while in the second one the diastasis associated to g is negative at some points. Thus, it is interesting to find examples of real analytic Kähler manifold (M, cg) which cannot be Kähler immersed into any (finite or infinite dimensional) complex space form for any $c > 0$ and satisfy:

- (i) the Kähler form ω associated to g is integral;
- (ii) the diastasis associated to g is globally defined on $M \times M$ and positive.

Our main result in this direction is the following theorem, proved at the end of next section:

Theorem 1. *Let $g = \frac{1}{1+|z|^2} dz \otimes d\bar{z}$ be the Cigar metric on \mathbb{C} . Then the diastasis function of the metric g is globally defined and positive on $\mathbb{C} \times \mathbb{C}$ and (\mathbb{C}, cg) cannot be (locally) Kähler immersed into any complex space form for any $c > 0$.*

Remark 2. It is worth pointing out that the cigar metric has positive sectional curvature. Hence, in view of Theorem 1, it is interesting to see if there exist examples of negatively curved real analytic Kähler manifolds (M, g) with globally defined diastasis function which is positive and such that (M, cg) cannot be locally Kähler immersed into any complex space form for all $c > 0$.

2. THE CIGAR METRIC ON \mathbb{C} AND THE PROOF OF THEOREM 1

The Cigar metric g on \mathbb{C} has been introduced by Hamilton in [8] as first example of Kähler–Ricci soliton on non-compact manifolds. It is defined by:

$$g = \frac{dz \otimes d\bar{z}}{1 + |z|^2}.$$

In [13] the authors of the present paper study Kähler–Ricci solitons, with the Cigar metric as particular case, from the symplectic point of view. A (globally defined) Kähler potential for this metric is given by (see also [14]):

$$D_0(|z|^2) = \int_0^{|z|^2} \frac{\log(1+s)}{s} ds,$$

whose power series expansion around the origin reads:

$$(2) \quad D_0(|z|^2) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{|z|^{2j}}{j^2}.$$

By duplicating the variable in this last expression, by (1) we get:

$$(3) \quad D^g(z, w) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} (|z|^{2j} + |w|^{2j} - (z\bar{w})^{2j} - (w\bar{z})^{2j}).$$

In the following lemma we prove that $D^g(z, w)$ is everywhere nonnegative and globally defined on $\mathbb{C} \times \mathbb{C}$. It is worth pointing out that the fact that $D^g(z, w)$ is globally defined was already observed in [14].

Lemma 3. *The diastasis function (3) of the Cigar metric is globally defined and nonnegative.*

Proof. If we denote by $\text{Li}_n(z)$ the polylogarithm function, defined for $|z| < 1$ by:

$$\text{Li}_n(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^n},$$

and by analytic continuation otherwise, from (3) we can write $D^g(z, w)$ as:

$$D(z, w) = -\text{Li}_2(-|z|^2) - \text{Li}_2(-|w|^2) + \text{Li}_2(-z\bar{w}) + \text{Li}_2(-w\bar{z}).$$

Write $z = \rho_1 e^{i\theta_1}$ and $w = \rho_2 e^{i\theta_2}$ and let $\alpha = \theta_1 - \theta_2$. Then:

$$\begin{aligned} (4) \quad D(z, w) &= -\text{Li}_2(-\rho_1^2) - \text{Li}_2(-\rho_2^2) + \text{Li}_2(-\rho_1 \rho_2 e^{i\alpha}) + \text{Li}_2(-\rho_1 \rho_2 e^{-i\alpha}) \\ &= -\text{Li}_2(-\rho_1^2) - \text{Li}_2(-\rho_2^2) + 2\text{ReLi}_2(-\rho_1 \rho_2 e^{i\alpha}), \end{aligned}$$

where we are allowed to take the real parts since $D^g(z, w)$ is real. In order to simplify the term $\text{ReLi}_2(-\rho_1 \rho_2 e^{i\alpha})$, we recall the following formula due to Kummer (see [9] or [10, p.15]):

$$\begin{aligned} \text{ReLi}_2(\rho e^{i\theta}) &= \frac{1}{2} \left(\text{Li}_2(\rho e^{i\theta}) + \overline{\text{Li}_2(\rho e^{i\theta})} \right) \\ &= -\frac{1}{2} \left(\int_0^\rho \frac{\log(1 - ye^{i\theta})}{y} dy + \int_0^\rho \frac{\log(1 - ye^{-i\theta})}{y} dy \right) \\ &= -\frac{1}{2} \int_0^\rho \frac{\log(1 - 2y \cos \theta + y^2)}{y} dy \end{aligned}$$

i.e.:

$$\text{ReLi}_2(-\rho e^{i\alpha}) = -\frac{1}{2} \int_0^\rho \frac{\log(1 + 2y \cos(\alpha) + y^2)}{y} dy.$$

Since $1 + 2y \cos(\alpha) + y^2$ is decreasing for $0 < \alpha < \pi$ and increasing for $\pi < \alpha < 2\pi$, $\alpha = \pi$ is a minimum. Thus:

$$(5) \quad \text{ReLi}_2(-\rho e^{i\alpha}) \geq -\int_0^\rho \frac{\log(|1 - y|)}{y} dy,$$

where:

$$-\int_0^\rho \frac{\log(|1 - y|)}{y} dy = \begin{cases} \text{Li}_2(\rho) & \text{if } \rho \leq 1 \\ \frac{\pi^2}{6} - \text{Li}_2(1 - \rho) - \ln(\rho - 1) \ln(\rho) & \text{otherwise.} \end{cases}$$

Thus, when $\rho_1 \rho_2 \leq 1$ from (4) and (5), we get:

$$D(z, w) \geq -\text{Li}_2(-\rho_1^2) - \text{Li}_2(-\rho_2^2) + 2\text{Li}_2(\rho_1 \rho_2) \geq 0,$$

where the last equality follows since all the factors in the sum are positive. When $\rho_1 \rho_2 > 1$, from (4) and (5), we get:

$$(6) \quad D(z, w) \geq -\text{Li}_2(-\rho_1^2) - \text{Li}_2(-\rho_2^2) + \frac{\pi^2}{3} - 2\text{Li}_2(1 - \rho_1 \rho_2) - 2 \ln(\rho_1 \rho_2 - 1) \ln(\rho_1 \rho_2).$$

The RHS is positive for $1 < \rho_1 \rho_2 \leq 2$ since it is sum of positive factors. When $\rho_1 \rho_2 > 2$, since all the factors are monotonic, it is enough to consider the limit as ρ_1 goes to $+\infty$ of $-D(z, w)/\text{Li}_2(-\rho_1^2)$. By (6) above we get:

$$\lim_{\rho_1 \rightarrow +\infty} \frac{D(z, w)}{-\text{Li}_2(-\rho_1^2)} \geq \frac{5}{2},$$

and we are done. \square

In order to prove Theorem 1 we need the following definition and properties of Bell polynomials. The partial Bell polynomials $B_{n,k}(x) := B_{n,k}(x_1, \dots, x_{n-k+1})$ of degree n and weight k are defined by (see e.g. [4, p. 133]):

$$(7) \quad B_{n,k}(x_1, \dots, x_{n-k+1}) = \sum_{\pi(k)} \frac{n!}{s_1! \dots s_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{s_1} \left(\frac{x_2}{2!}\right)^{s_2} \dots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{s_{n-k+1}},$$

where the sum is taken over the integers solutions of:

$$\begin{cases} s_1 + 2s_2 + \dots + ks_{n-k+1} = n \\ s_1 + \dots + s_{n-k+1} = k. \end{cases}$$

Bell polynomials satisfy the following equalities, which are fundamental for the proof of our result (the second one has been firstly pointed out in [5]):

$$(8) \quad B_{n,k}(trx_1, tr^2x_2, \dots, tr^{n-k+1}x_{n-k+1}) = t^k r^n B_{n,k}(x_1, \dots, x_{n-k+1}).$$

$$(9) \quad B_{n,k+1}(x) = \frac{1}{(k+1)!} \underbrace{\sum_{\alpha_1=k}^{n-1} \sum_{\alpha_2=k-1}^{\alpha_1-1} \dots \sum_{\alpha_k}^{\alpha_{k-1}-1}}_k \overbrace{\binom{n}{\alpha_1} \binom{\alpha_1}{\alpha_2} \dots \binom{\alpha_{k-1}}{\alpha_k}}^k \cdot x_{n-\alpha_1} x_{\alpha_1-\alpha_2} \dots x_{\alpha_{k-1}-\alpha_k} x_{\alpha_k}.$$

The complete Bell polynomials are given by:

$$Y_n(x_1, \dots, x_n) = \sum_{k=1}^n B_{n,k}(x), \quad Y_0 := 0,$$

and the role they play in our context is given by the following formula [4, Eq. 3b, p.134]:

$$(10) \quad \frac{d^n}{dx^n} \left(\exp \left(\sum_{j=1}^{\infty} a_j \frac{x^j}{j!} \right) \right) \Big|_0 = Y_n(a_1, \dots, a_n).$$

Observe that from (8) it follows:

$$(11) \quad Y_n(rx_1, r^2x_2, \dots, r^nx_n) = r^n Y_n(x_1, \dots, x_n).$$

The following lemma has a key role in the proof of Theorem 1.

Lemma 4. *Let $a_j = j!/j^2$. Then*

$$\lim_{n \rightarrow \infty} \frac{(2n)^2 B_{2n,k+1}(a)}{(2n)!} = \frac{k+1}{(k+1)!} \sum_{j_1=1}^{\infty} \frac{1}{j_1^2} \cdots \sum_{j_{k-1}=1}^{\infty} \frac{1}{j_{k-1}^2}.$$

Proof. Observe first that by (9), we get:

$$(12) \quad \frac{B_{2n,k+1}(a)}{(2n)!} = \frac{1}{(k+1)!} \underbrace{\sum_{\alpha_1=k}^{2n-1} \sum_{\alpha_2=k-1}^{\alpha_1-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_k \frac{1}{(2n-\alpha_1)^2 (\alpha_1-\alpha_2)^2 \cdots (\alpha_{k-1}-\alpha_k)^2 \alpha_k^2}.$$

We proceed by induction on k . For $k=1$ we have:

$$\begin{aligned} \frac{(2n)^2 B_{2n,2}(\tilde{a})}{(2n)!} &= \frac{(2n)^2}{2} \sum_{\alpha_1=1}^{2n-1} \frac{1}{(2n-\alpha_1)^2 \alpha_1^2} \\ &= \frac{1}{2} \left(\frac{(2n)^2}{(2n-1)^2} + \frac{(2n)^2}{(2n-2)^2 2^2} + \frac{(2n)^2}{(2n-3)^2 3^2} + \cdots + \frac{(2n)^2}{2^2 (2n-2)^2} + \frac{(2n)^2}{(2n-1)^2} \right) \\ &= \frac{(2n)^2}{(2n-1)^2} + \frac{(2n)^2}{(2n-2)^2 2^2} + \frac{(2n)^2}{(2n-3)^2 3^2} + \cdots + \frac{(2n)^2}{(n-1)^2 (n+1)^2} \\ &= \frac{(2n)^2}{(2n-1)^2} \sum_{j=1}^{\infty} \frac{1}{j^2} + \psi(n), \end{aligned}$$

where $\psi(n)$ satisfies $\lim_{n \rightarrow \infty} \psi(n) = 0$. Thus we have:

$$\lim_{n \rightarrow \infty} \frac{(2n)^2 B_{2n,2}(\tilde{a})}{(2n)!} = \sum_{j=1}^{\infty} \frac{1}{j^2}.$$

Assume now that:

$$(13) \quad \lim_{n \rightarrow \infty} \frac{(2n)^2 B_{2n,k}(\tilde{a})}{(2n)!} = \frac{k}{k!} \sum_{j_1=1}^{\infty} \frac{1}{j_1^2} \cdots \sum_{j_{k-1}=1}^{\infty} \frac{1}{j_{k-1}^2}.$$

By (12) this implies:

$$\lim_{n \rightarrow \infty} \underbrace{\sum_{\alpha_1=k-1}^{2n-1} \sum_{\alpha_2=k-2}^{\alpha_1-1} \cdots \sum_{\alpha_{k-1}=1}^{\alpha_{k-2}-1}}_{k-1} \frac{(2n)^2}{(2n-\alpha_1)^2 (\alpha_1-\alpha_2)^2 \cdots (\alpha_{k-2}-\alpha_{k-1})^2 \alpha_{k-1}^2} = k \sum_{j_1=1}^{\infty} \frac{1}{j_1^2} \cdots \sum_{j_{k-1}=1}^{\infty} \frac{1}{j_{k-1}^2}$$

and thus:

$$\begin{aligned}
\frac{(2n)^2 B_{2n,k+1}(\tilde{a})}{(2n)!} &= \frac{(2n)^2}{(k+1)!} \underbrace{\sum_{\alpha_1=k}^{2n-1} \sum_{\alpha_2=k-1}^{\alpha_1-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_k \frac{1}{(2n-\alpha_1)^2 (\alpha_1-\alpha_2)^2 \cdots (\alpha_{k-1}-\alpha_k)^2 \alpha_k^2} \\
&= \frac{(2n)^2}{(k+1)!(2n-k)^2} \underbrace{\sum_{\alpha_2=k-1}^{k-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_{k-1} \frac{1}{(k-\alpha_2)^2 \cdots (\alpha_{k-1}-\alpha_k)^2 \alpha_k^2} + \\
&\quad + \frac{(2n)^2}{(k+1)!(2n-k-1)^2} \underbrace{\sum_{\alpha_2=k-1}^k \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_{k-1} \frac{1}{(k+1-\alpha_2)^2 \cdots (\alpha_{k-1}-\alpha_k)^2 \alpha_k^2} + \cdots \\
&\quad \cdots + \frac{(2n)^2}{(k+1)!n^2} \underbrace{\sum_{\alpha_2=k-1}^{n-1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_{k-1} \frac{1}{(n-\alpha_2)^2 \cdots (\alpha_{k-1}-\alpha_k)^2 \alpha_k^2} + \cdots \\
&\quad \cdots + \frac{1}{(k+1)!2^2} \underbrace{\sum_{\alpha_2=k-1}^{2n-3} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_{k-1} \frac{(2n)^2}{(2n-2-\alpha_2)^2 \cdots (\alpha_{k-1}-\alpha_k)^2 \alpha_k^2} \\
&\quad \frac{1}{(k+1)!} \underbrace{\sum_{\alpha_2=k-1}^{2n-2} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_{k-1} \frac{(2n)^2}{(2n-1-\alpha_2)^2 \cdots (\alpha_{k-1}-\alpha_k)^2 \alpha_k^2} \\
&= \frac{(2n)^2}{(k+1)!(2n-k)^2} + \\
&\quad + \frac{(2n)^2}{(k+1)!(2n-k-1)^2} \underbrace{\sum_{\alpha_2=k-1}^k \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_{k-1} \frac{1}{(k+1-\alpha_2)^2 \cdots (\alpha_{k-1}-\alpha_k)^2 \alpha_k^2} + \\
&\quad + \frac{(2n)^2}{(k+1)!(2n-k-2)^2} \underbrace{\sum_{\alpha_2=k-1}^{k+1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_{k-1} \frac{1}{(k+2-\alpha_2)^2 \cdots (\alpha_{k-1}-\alpha_k)^2 \alpha_k^2} + \\
&\quad + \cdots + \Psi(n) + \cdots + \frac{k}{(k+1)!} \sum_{j_1=1}^{\infty} \frac{1}{j_1^2} \cdots \sum_{j_{k-1}=1}^{\infty} \frac{1}{j_k^2}
\end{aligned}$$

where $\Psi(n)$ is an infinitesimal function of n .

Then by (13):

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(2n)^2 B_{2n,k+1}(\tilde{a})}{(2n)!} &= \frac{1}{(k+1)!} + \frac{1}{(k+1)!} \underbrace{\sum_{\alpha_2=k-1}^k \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_{k-1} \frac{1}{(k+1-\alpha_2)^2 \cdots (\alpha_{k-1}-\alpha_k)^2 \alpha_k^2} + \\ &+ \frac{1}{(k+1)!} \underbrace{\sum_{\alpha_2=k-1}^{k+1} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_{k-1} \frac{1}{(k+2-\alpha_2)^2 \cdots (\alpha_{k-1}-\alpha_k)^2 \alpha_k^2} + \\ &+ \cdots + \frac{k}{(k+1)!} \sum_{j_1=1}^{\infty} \frac{1}{j_1^2} \cdots \sum_{j_{k-1}=1}^{\infty} \frac{1}{j_k^2} \end{aligned}$$

i.e.:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(2n)^2 B_{2n,k+1}(\tilde{a})}{(2n)!} &= \frac{1}{(k+1)!} \sum_{j=1}^{\infty} \underbrace{\sum_{\alpha_2=k-1}^{k-2+j} \cdots \sum_{\alpha_k=1}^{\alpha_{k-1}-1}}_{k-1} \frac{1}{(k-1+j-\alpha_2)^2 \cdots (\alpha_{k-1}-\alpha_k)^2 \alpha_k^2} + \\ &+ \frac{k}{(k+1)!} \sum_{j_1=1}^{\infty} \frac{1}{j_1^2} \cdots \sum_{j_{k-1}=1}^{\infty} \frac{1}{j_k^2}. \end{aligned}$$

By setting:

$$A_1(k) := 1, \quad A_j(k) = \frac{k}{j^2} + k \sum_{s=2}^{j-1} \frac{A_{j-s+1}(k-1)}{s^2},$$

we finally get:

$$\lim_{n \rightarrow \infty} \frac{(2n)^2 B_{2n,k+1}(\tilde{a})}{(2n)!} = \frac{1}{(k+1)!} \sum_{j=1}^{\infty} A_j(k) + \frac{k}{(k+1)!} \sum_{j_1=1}^{\infty} \frac{1}{j_1^2} \cdots \sum_{j_{k-1}=1}^{\infty} \frac{1}{j_k^2},$$

and conclusion follows by observing that:

$$\begin{aligned} \sum_{j=1}^{\infty} A_j(k) &= \sum_{j=1}^{\infty} \frac{k}{j^2} + A_2(k-1) \sum_{j=2}^{\infty} \frac{k}{j^2} + A_3(k-1) \sum_{j=3}^{\infty} \frac{k}{j^2} + \cdots \\ &= \sum_{j_1=1}^{\infty} \frac{1}{j_1^2} \cdots \sum_{j_{k-1}=1}^{\infty} \frac{1}{j_k^2}. \end{aligned}$$

□

We are now able to prove our main result.

Proof of Theorem 1. Observe first that if (M, cg) does not admit a Kähler immersion into \mathbb{CP}^∞ for any value of $c > 0$, then it does not either in any other space form. In fact,

if (M, cg) admits a Kähler immersion into $\ell^2(\mathbb{C})$ then by a theorem of Bochner (see [2]), it also does into \mathbb{CP}^∞ , and in particular since the multiplication by c is harmless when one considers Kähler immersion into flat spaces, it does for any value of $c > 0$. Further, in a totally similar way as in the proof of Bochner's statement, one can prove that a Kähler manifold admitting a Kähler immersion into \mathbb{CH}^∞ can also be Kähler immersed into $\ell^2(\mathbb{C})$. Thus, it is enough to show that (\mathbb{C}, cg) does not admit a Kähler immersion into \mathbb{CP}^∞ for any $c > 0$.

Further, the diastasis function associated to the Cigar metric is globally defined and positive by Lemma 3 above.

Then, by Calabi's criterion stated above, it remains only to show that there exists n such that:

$$\frac{\partial^{2n} \exp(cD_0(|z|^2))}{\partial z^n \partial \bar{z}^n} \Big|_0 < 0,$$

where $D_0(|z|^2)$ is the Kähler potential defined in (2). Observe first that setting:

$$(14) \quad \tilde{a}_j := -c \frac{j!}{j^2},$$

by (10) and (11) we get:

$$\begin{aligned} \frac{\partial^{2n} \exp(cD_0(|z|^2))}{\partial z^n \partial \bar{z}^n} \Big|_0 &= \frac{1}{n!} \frac{d^n \exp(cD_0(x))}{dx^n} \Big|_0 \\ &= \frac{1}{n!} Y_n(-\tilde{a}_1, (-1)^2 \tilde{a}_2, \dots, (-1)^n \tilde{a}_n) \\ &= \frac{(-1)^n}{n!} Y_n(\tilde{a}_1, \dots, \tilde{a}_n). \end{aligned}$$

We wish to prove that for any $c > 0$ there exists n big enough such that:

$$Y_{2n}(a_1, \dots, a_{2n}) < 0.$$

Observe first that since $\tilde{a}_j = -c a_j$ with $a_j = j!/j^2$, we get:

$$\begin{aligned} Y_{2n}(\tilde{a}) &= \sum_{k=1}^{2n} (-1)^k c^k B_{2n,k}(a) \\ &= \frac{(2n)!c}{(2n)^2} \left(-1 + \frac{c(2n)^2 B_{2n,2}(a)}{(2n)!} - \frac{c^2(2n)^2 B_{2n,3}(a)}{(2n)!} + \dots + \frac{c^{2n-1}(2n)^2}{(2n)!} \right). \end{aligned}$$

Thus, we need to prove that for any value of c there exists n large enough such that the following inequality holds:

$$(15) \quad \frac{c(2n)^2 B_{2n,2}(a)}{(2n)!} - \frac{c^2(2n)^2 B_{2n,3}(a)}{(2n)!} + \dots + \frac{c^{2n-1}(2n)^2}{(2n)!} < 1.$$

By Lemma 4,

$$\lim_{n \rightarrow +\infty} \frac{(2n)^2 B_{2n,k+1}(a)}{(2n)!} = \frac{k+1}{(k+1)!} \sum_{j_1=1}^{\infty} \frac{1}{j_1^2} \sum_{j_2=1}^{\infty} \frac{1}{j_2^2} \cdots \sum_{j_k=1}^{\infty} \frac{1}{j_k^2},$$

and since:

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6},$$

we get:

$$\lim_{n \rightarrow +\infty} \frac{(2n)^2 B_{2n,k+1}(a)}{(2n)!} = \frac{1}{k!} \left(\frac{\pi^2}{6} \right)^k.$$

Plugging this into (15), we get that as n goes to infinity the left hand side converge to:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} c^k}{k!} \left(\frac{\pi^2}{6} \right)^k = 1 - e^{-\frac{c\pi^2}{6}},$$

and conclusion follows by observing that $1 - e^{-\frac{c\pi^2}{6}}$ is strictly increasing as a function on c and its limit value as c grows is 1. \square

Remark 5. With similar tools one can prove that (\mathbb{C}, cg) , namely the complex plane with a multiple of the Cigar metric does not admit a Kähler immersion into any indefinite flat space of finite signature.

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